A GENERALIZATION OF RADON'S THEOREM

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1. Let "m-set" mean a set of m points in R^n. We shall say that an m-set is r-divisible if it can be divided into r sets in such a way that the convex hulls of the r sets have a non-empty intersection. B. Birch [**1**] conjectured

THEOREM 1. *Any* $\big(r(n+1)-n\big)$*-set is r-divisible,*

and proved it in the case $n=2$. In the case $n>2$, Birch proved a weaker result, with $r(n+1)-n$ replaced by $rn(n+1)-n^2-n+1$. This was, for most r and n, an improvement of the earlier result, by R. Rado [**2**], that any $\big((r-2)\,2^n+n+2\big)$-set is r-divisible.

The case $r=2$ was proved by J. Radon [**3**], and used by him for proving Helly's theorem. The reader is referred to [**4**] for a discussion of Radon's and Helly's theorems and related questions.

In order to see that Theorem 1 is best possible, *i.e.* that some (in fact almost all) $\big(r(n+1)-n-1\big)$-sets are not r-divisible we consider an $\big(r(n+1)-n-1\big)$-set Ω, the points of which are algebraically independent, and a partition of Ω into sets $\Omega_1, \ldots, \Omega_r$. (We say that m points are algebraically independent if their coordinates are mn real numbers, algebraically independent over the field of rational numbers.) It suffices to show that the intersection of $L_1, \ldots, L_r$, the *linear* hulls of $\Omega_1, \ldots, \Omega_r$, is empty. Hence assume that $L_1 \cap \ldots \cap L_r \neq \phi$. This is a purely algebraic property of Ω and the given partition of Ω, and so, by the algebraic independence in Ω, we conclude that whenever sets $\Omega_1', \ldots, \Omega_r'$ are given, with linear hulls $L_1', \ldots, L_r'$, then $L_1' \cap \ldots \cap L_r' \neq \phi$, provided, for each i, Ω_i' is equipollent to Ω_i and L_i' has the same dimension as L_i. (Strictly speaking, this statement is correct only if interpreted in real *projective* n-space.)

One now gets the desired contradiction by choosing first r non-intersecting (also at infinity) linear spaces $L_1', \ldots, L_r'$ such that, for each i, $\dim L_i' = \dim L_i$, and then, in each L_i', a set Ω_i', equipollent to Ω_i, the linear hull of which is L_i'. The feasibility of this is granted by the inequality

$$\begin{aligned}\operatorname{codim} L_1 + \ldots + \operatorname{codim} L_r &\geqslant \big(n+1-(\text{number of points in } \Omega_1)\big)+\ldots \\ &= r(n+1)-\big(r(n+1)-n-1\big) = n+1.\end{aligned}$$

Below we shall give a proof of Theorem 1 in its full generality. The author would like to thank Birch and Rado for stimulating discussions on a very early version of this paper.

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2. We first prove three lemmas.

LEMMA 1. *Let $V_1, \ldots, V_s$ be linear subspaces of n-space, none of which contains a given point A. Let W_i be the space spanned by V_i and A and assume that, for every i, W_i intersects $V_1 \cap \ldots \cap V_{i-1} \cap V_{i+1} \cap \ldots \cap V_s$ in a single point B_i. Assume furthermore that* $\operatorname{codim} V_1 + \ldots + \operatorname{codim} V_s = n+1$. *Then, for some i, V_i does not separate A and B_i.*

We start with the case when $\operatorname{codim} V_1 = \ldots = \operatorname{codim} V_s = 1$. By our assumptions s then equals $n+1$, and each space W_i equals the full n-space. Thus each point B_i is the intersection of the hyperplanes V_j, $j \neq i$. We may assume the points $B_1, \ldots, B_{n+1}$ to be linearly independent, so that they form the basis of a barycentric coordinate system in n-space. (If B_1, say, is in the linear hull of $B_2, \ldots, B_{n+1}$, B_1 belongs to V_1, as $B_2, \ldots, B_{n+1}$ are all in V_1, but then V_1 does not separate A and B_1, and the lemma holds.) In this system A has $n+1$ coordinates, the sum of which equals 1. Thus some coordinate, say the first one, must be positive. This means that A and B_1 are not separated by the hyperplane that is spanned by $B_2, \ldots, B_{n+1}$. This latter hyperplane is, however, identical to V_1.

If, say, $\operatorname{codim} V_1 > 1$, then W_1 is a proper subspace of n-space. Reasoning by induction, we may thus assume that Lemma 1 holds in W_1. We apply it to $V_1 \cap W_1, \ldots, V_s \cap W_1$ and A. As the modular law holds in the lattice of all linear subspaces of a linear space, we see that the space spanned by $V_i \cap W_1$ and A equals the intersection with W_1 of the space spanned by V_i and A, *i.e.* of W_i. Now we have

$$\{(V_1 \cap W_1) \cap \ldots \cap (V_{i-1} \cap W_1) \cap (V_{i+1} \cap W_1) \cap \ldots \cap (V_s \cap W_1)\} \cap (W_i \cap W_1) = B_i.$$

Thus it only remains to compute the sum of the codimensions in W_1 of $V_1 \cap W_1, \ldots, V_s \cap W_1$. If $i > 1$, then

$$\begin{aligned} n = \operatorname{codim} B_1 &\leqslant \operatorname{codim} V_i \cap W_1 + \operatorname{codim} V_2 + \ldots + \operatorname{codim} V_{i-1} + \operatorname{codim} V_{i+1} + \ldots + \operatorname{codim} V_s \\ &= \operatorname{codim} V_i \cap W_1 + n - \operatorname{codim} W_1 - \operatorname{codim} V_i \leqslant n. \end{aligned}$$

Hence $\operatorname{codim} V_i \cap W_1 = \operatorname{codim} W_1 + \operatorname{codim} V_i$, which means that the codimension of $V_i \cap W_1$ in W_1 equals $\operatorname{codim} V_i$. The sum of the codimensions in W_1 of $V_1 \cap W_1, \ldots, V_s \cap W_1$ thus equals

$$1 + (n + 1 - \operatorname{codim} V_1) = 1 + \dim W_1.$$

We conclude that, for some i, $V_i \cap W_1$ does not separate A and B_i. But then V_i does not separate A and B_i.

LEMMA 2. *An m-set Ω that is the limit of a sequence $\Omega_1, \Omega_2, \ldots$ of r-divisible m-sets, is itself r-divisible.*

We put $\Omega_j = \{P_{j1}, \ldots, P_{jm}\}$, $\Omega = \{P_1, \ldots, P_m\}$. Our assumptions are that

$$\lim_{j\to\infty} P_{jk} = P_k, \quad k = 1, \ldots, m, \tag{1}$$

and that for each j there exists a partition of the set $\{1, \ldots, m\}$ such that the convex hulls of the sets obtained by partitioning Ω_j in the corresponding way, have a non-empty intersection. Assume that for each j we have chosen a point R_j in that intersection. Then there must be a partition of $\{1, \ldots, m\}$ into sets $\Gamma_1, \ldots, \Gamma_r$ such that, for infinitely many j,

$$R_j \in \text{convex hull } (\{P_{jk} \mid k \in \Gamma_i\}), \quad i = 1, \ldots, r. \tag{2}$$

We may as well assume that (2) holds for *all* j, as no harm is done by replacing the originally given sequence of m-sets by a subsequence. The relations (1) and (2) show that the sequence $R_1, R_2, \ldots$ is bounded, hence contains a convergent subsequence. We may as well assume $R_1, R_2, \ldots$ itself to be convergent, towards a point R. Then, by (1) and (2),

$$R \in \text{convex hull } (\{P_k \mid k \in \Gamma_i\}), \quad i = 1, \ldots, r,$$

which proves the lemma.

LEMMA 3. *Let $Q, P_1, \ldots, P_N$ $\left(N = r(n+1) - n\right)$ be algebraically independent points. Then $\{Q, P_2, \ldots, P_N\}$ is r-divisible if $\{P_1, \ldots, P_N\}$ is r-divisible.*

For each real number t, put $\Omega(t) = \{(1-t)P_1 + tQ, P_2, \ldots, P_N\}$. We assume that $\Omega(0)$ is r-divisible, and we shall prove that $\Omega(1)$ is r-divisible. We do this by proving that the set

$$T = \{t \mid \Omega(t) \text{ is } r\text{-divisible}\}$$

is both open and closed. T, being non-empty, must then consist of all real numbers; in particular T must contain the number 1.

By Lemma 2, T is closed; hence it remains to prove that T is open. Let now t_0 be an arbitrary point of T. We make a study, first of the set $\Omega(t_0)$, and then of the sets $\Omega(t)$ when t is near t_0.

As $t_0 \in T$, there is a point L and a partition of $\Omega(t_0)$ into sets

$$\Omega_1 = \{(1-t_0)P_1 + t_0 Q, P_2, \ldots, P_{n_1+1}\}, \ldots, \Omega_r$$

such that L is in the convex hull of each Ω_i. If some Ω_i consists of more than $n+1$ points, there is a subset Ω_i' of Ω_i, containing only $n+1$ points and having L in its convex hull, by Carathéodory's theorem (see [4]). This means that we may assume each Ω_i to contain $n_i + 1$ points, $n_i \leqslant n$, because $N < r(n+1) + 1$.

Now, when $i > 1$, the convex hull of Ω_i is a non-degenerate n_i-simplex σ_i, by the algebraic independence of the points in Q_i. L_i, the linear hull of Ω_i, is thus an n_i-space. If L_1 is not an n_1-space, *i.e.* if σ_1 is a degenerate

n_1-simplex, L_1 must be the (n_1-1)-space H that is spanned by the n_1 algebraically independent points $P_2, \ldots, P_{n_1+1}$. Then the point $(1-t_0)P_1+t_0 Q$ is in H, and we get

$$L \in \sigma_1 \cap \ldots \cap \sigma_r \subset H \cap L_2 \cap \ldots \cap L_r.$$

But the linear spaces $H, L_2, \ldots, L_r$ are algebraically independent and the sum of their codimensions equals

$$rn-\Big((n_1-1)+n_2+\ldots+n_r\Big)=rn+1-(N-r)=n+1.$$

This is a contradiction, and we conclude that also σ_1 is non-degenerate.

Let $\Omega_1(t)=\{(1-t)P_1+tQ, P_2, \ldots, P_{n_1+1}\}$. Then there is an open interval $I_1 \ni t_0$, such that when $t \in I_1$, $\Omega_1(t)$ has a convex hull $\sigma_1(t)$ which is a non-degenerate n_1-simplex, with linear hull $L_1(t)$. Let $L(t)$ denote the space $L_1(t) \cap L_2 \cap \ldots \cap L_r$. What can be said about $L(t)$? If K is the linear hull of $\{Q, P_1, \ldots, P_{n_1+1}\}$, then $L_1(t) \subset K$ and, accordingly, $L(t) \subset K \cap L_2 \cap \ldots \cap L_r$ for all values of t. If $n_1=n$ K is all of n-space and the sum of the codimensions of $L_2, \ldots, L_r$ equals n. Thus $L(t)$ is a single point, which is independent of t. If $n_1<n$, K is an (n_1+1)-space, algebraically independent of $L_2, \ldots, L_r$. Thus $K \cap L_2 \cap \ldots \cap L_r$ is a 1-space, a line M. This means that $L(t_0)$, if it is not the single point L, equals M. Now $L(0) \in M$, and hence, if $L(t_0)=M$, $L(0) \in L_1(t_0) \cap L_1(0)$.† Further, $L(0)$ is not in H, the linear hull of $\{P_2, \ldots, P_{n_1+1}\}$, as we have seen that $H \cap L_2 \cap \ldots \cap L_r$ is empty. This shows that the space $L_1(t_0) \cap L_1(0)$, which clearly contains H, must contain H strictly, *i.e.* $L_1(t_0)=L_1(0)$. Similarly, we find that $L_1(t_0)=L_1(1)$. But then $Q \in L_1(1)=L_1(0)$, which is impossible, as Q is algebraically independent of the points in $\Omega_1(0)$ (remember that $n_1<n$).

Hence $L(t_0)$ consists of the point L only. Furthermore, there is an open interval $I_2 \subset I_1$, with $t_0 \in I_2$, such that, for all t in I_2, $L(t)$ is a single point, depending continuously on t. Actually,

$$L(t)=\alpha(1-t)\Big(\alpha(1-t)+\beta t\Big)^{-1}L(0)+\beta t\Big(\alpha(1-t)+\beta t\Big)^{-1}L(1),$$

where β is the barycentric coordinate of $L(0)$ with respect to P_1 in the system with basis $\Omega_1(0)$ and α is the barycentric coordinate of $L(1)$ with respect to Q in the system with basis $\Omega_1(1)$.

Let us prove that t_0 is an interior point of T. The easier case is when L is in the interior of each of the simplices $\sigma_1, \ldots, \sigma_r$. Then, by continuity, t_0 belongs to an open interval $I_3 \subset I_2$, such that when $t \in I_3$,

$$L(t) \in \sigma_1(t) \cap \sigma_2 \cap \ldots \cap \sigma_r.$$

Thus, in this case, $\Omega(t)$ is r-divisible when $t \in I_3$. Assume now that L is on

† Note that $L(0)$ is a single point, because $L_1(0), L_2, \ldots, L_r$ are algebraically independent spaces, the sum of the codimensions of which equals n. Likewise $L(1)$ is a single point.

the boundary of one of the simplices, *e.g.* σ_a. Then L is on some (n_a-1)-face of σ_a, the opposite vertex of which is, say, P_b. [Note that, if $a=1$, L cannot be on that face of σ_1 which is opposite to $(1-t_0)P_1+t_0Q$, as we have seen that $H\cap L_2\cap\ldots\cap L_r$ is empty.] The number b, which also determines a, of course, is uniquely determined as we can see in the following way. Assume, namely, that L is also on some other face, the one opposite P_c, say, of some simplex σ_d (where maybe $d=a$). We then look at the sets which are obtained from $\{Q, P_1, \ldots, P_{n_1+1}\}$, $\Omega_2, \ldots, \Omega_r$ by taking away P_b and P_c. The sum of the codimensions of the linear hulls of these sets equals $n+1$ ($n+2$ if $n_1=n$ and $b>n+1$, $c>n+1$), and so, by algebraic independence, they do not intersect, whereas we have assumed them all to contain L. This contradiction shows the uniqueness of b, so that not only does L belong to the boundary of exactly one simplex, namely σ_a, but L is also in exactly one (n_a-1)-face, let us call it τ_b, of σ_a. Thus L is an inner point of τ_b. This can also be expressed as follows: In the space L_a, σ_a is the intersection of n_a+1 closed half-spaces. L belongs to the interior of all but one of these. The remaining one has L on its boundary, which is the hyperplane π_b spanned by τ_b.

By continuity, the results above yield the existence of a neighbourhood I_4, of t_0, $I_4\subset I_2$, such that, when $t\in I_4$, the following is true. $L(t)$ is in each of the simplexes $\sigma_1(t), \sigma_2, \ldots, \sigma_r$, except possibly $\sigma_a\big(\sigma_1(t)$ if $a=1\big)$. $L(t)$ is in each of the half-spaces whose intersection is $\sigma_a\big(\sigma_1(t)$ if $a=1\big)$, except possibly the one that has P_b in its interior.

This means that for each t in I_4, a sufficient condition for $\Omega(t)$ to be r-divisible is that the space $\pi_b(t)$ (the one spanned by $\Omega_a-\{P_b\}$ if $a>1$ and by $\Omega_1(t)-\{P_b\}$ if $a=1$) does not separate $L(t)$ and P_b.

Till now we have only been considering one special partition of $\Omega(t_0)$.

There are, however, certain other partitions that are worthy of consideration. Namely, let x $(\neq a)$ be such that $n_x<n$. Then each of the sets $\Omega_1, \ldots, \Omega_x\cup\{P_b\}, \ldots, \Omega_a-\{P_b\}, \ldots, \Omega_r$ contains less than $n+2$ points, and the convex hulls of these sets all contain the point L. L is on the boundary of one of these hulls, namely that of $\Omega_x\cup\{P_b\}$. This allows us to conclude that there is a neighbourhood I_4^x of t_0, such that, for each t in I_4^x a sufficient condition for $\Omega(t)$ to be r-divisible is that the space L_x $\big(L_1(t)$ if $x=1\big)$ does not separate $L^x(t)$ and P_b. Here $L^x(t)$ is the intersection of the linear hulls of the sets obtained from

$$\Omega_1, \ldots, \Omega_x\cup\{P_b\}, \ldots, \Omega_a-\{P_b\}, \ldots, \Omega_r$$

on replacing the point $(1-t_0)P_1+t_0Q$ by the point $(1-t)P_1+tQ$.

Let now I_5 be the intersection between I_4 and the neighbourhoods $I_4^{x_1}, I_4^{x_2}, \ldots$. Then, if t is in I_5, we can apply Lemma 1 to the spaces $\pi_b(t), L_{x_1}, L_{x_2}, \ldots$ if $n_1=n$ or $a=1$, or to $\pi_b(t), L_1(t), L_{x_2}, \ldots$ if $n_1<n$ and $a>1$, and to the point P_b. It is clear that the conditions of the lemma are satisfied, the points B_i being the points $L(t)$, $L^{x_1}(t)$, $L^{x_2}(t)$, … . The conclusion of the lemma then states that at least one of our sufficient conditions for $\Omega(t)$ to be r-divisible is satisfied, and Lemma 3 is proved.

Theorem 1 itself is now an almost immediate consequence of the Lemmas 2 and 3. We choose an r-divisible N-set Ω_1, $\bigl(N=r(n+1)-n\bigr)$, the points of which are algebraically independent. We may, for instance, let Ω_1 consist of a point and the vertices of $r-1$ n-simplices containing that point. If an N-set Ω is given, we can find a sequence Ω_1, Ω_2, … of N-sets, converging towards Ω, and having the property that $\Omega_i \cup \Omega_{i+1}$ is, for all i, an $(N+1)$-set of algebraically independent points. By Lemma 3, and the r-divisibility of Ω_1, Ω_2 is r-divisible. Lemma 3, applied once more, then shows that Ω_3 is r-divisible, etc., whereupon Theorem 1 follows by Lemma 2.

THEOREM 2. *For any $\bigl(r(n+1)-n\bigr)$-set Ω there exists a point R such that any closed half-space containing R contains at least r points from Ω.*

This theorem, which is a special case of a theorem by R. Rado [**5**], deserves its place here. Indeed, it was the prospect of giving the following transparent proof of Theorem 2 that first made the author conjecture and try to prove Theorem 1.

By Theorem 1, there are disjoint sets Ω_1, …, Ω_r, the union of which is Ω, and there is a point R that belongs to the convex hull of each set Ω_i. Any closed half-space containing R will then contain at least one point from each set Ω_i, and thus at least r points from Ω.

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